

Unusual decoherence in qubit measurements with a Bose-Einstein condensate

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We consider an electrostatic qubit located near a Bose-Einstein condensate (BEC) of noninteracting bosons in a double-well potential, which is used for qubit measurements. Tracing out the BEC variables we obtain a simple analytical expression for the qubit's density-matrix. The qubit's evolution exhibits a slow ($\propto 1/\sqrt{t}$) damping of the qubit's coherence term, which however turns to be a Gaussian one in the case of static qubit. This stays in contrast to the exponential damping produced by most classical detectors. The decoherence is, in general, incomplete and strongly depends on the initial state of the qubit.

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I. INTRODUCTION

Recent progress in quantum information technology has lead to significant technological and theoretical advances in measuring and controlling the state of a two-level quantum system (qubit). Devices used for this purpose include point-contact detectors and single electron transistors [1, 2, 3] where the magnitude of electron current is used to determine the qubit's state. Recently, more sophisticated hybrid systems which combine a charged qubit with microwave resonators or ensembles cold polar molecules were proposed [4, 5]. In addition to technological benefits such hybrids offer an insight into fundamental physical phenomena, such as decoherence. The decoherence is present for any microscopic system (e.g., a qubit) interacting with a macroscopic device, characterized by a large number of degrees of freedom and a dense distribution of energy levels. As a result, an initial state of a qubit is expected to be rapidly (exponentially in time) converted into a statistical mixture, so that the information stored in the qubit is erased [6]. For example, this has been explicitly demonstrated for qubit measurements with a point-contact detector shown in Fig.1a, where a macroscopic current flowing into the right reservoir across the potential barrier is modulated by the qubit's electron [7].

In this letter we study measurements in a hybrid system, consisting of an electrostatic qubit placed in closed proximity to a non-interacting BEC trapped in a symmetric double-well potential. The qubit is represented by an electron in coupled quantum dots (Fig.1b), while confinement of the BEC can be realized, for example, by means of a quasi-electrostatic optical dipole trap produced by two crossed laser beams [8]. Since the trapping occurs due the interaction of the induced atomic dipole

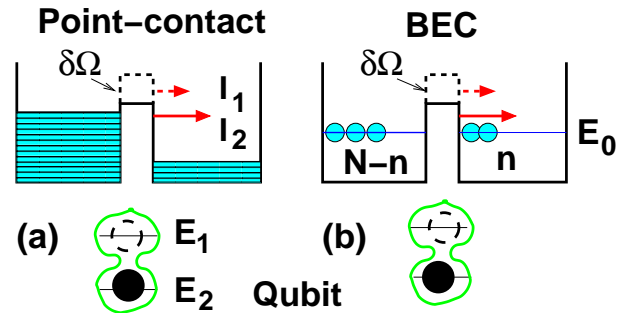


FIG. 1: (Color on line) a) Point-contact in which electrons, initially in the left reservoir, tunnel across a potential barrier modulated by the presence of an electron in one of the two coupled quantum dots (qubit); b) Atoms of a BE condensate tunnel across a barrier of a symmetric double-well potential modulated by a qubit.

moment of neutral atoms and the far-detuned optical field [9], additional electric field induced by the electron would change the barrier height. This, in turn, would modulate the atomic current just as the presence of an electron in one of the dots modifies the current of the point-contact detector shown in Fig.1a. There is, however, an important difference as only a single level (zero-width band) is available for the tunneling atoms, which raises the question of what type of decoherence, if any, would experience the measured qubit?

With the number of carriers macroscopically large, but only one level existing in each of the reservoirs, the question cannot be answered without a detailed analysis. On one hand, the bosons are moving independently, and one could expect their effect to be similar to that of a single boson which, as is easy to show, does not produce decoherence. On the other hand, it is not clear whether the large number of uncorrelated degrees of freedom in the detector will not have an averaging effect on the qubit thus causing the off-diagonal elements of its density matrix to disappear.

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In the following we will show that the density matrix of a qubit coupled to a BEC will undergo an evolution which is not described by Bloch-like equations similar to those arising in the case of a point-contact detector [7]. However, the BEC model shown in Fig.1b can be solved exactly. We will demonstrate that the rate of decoherence is extremely (non-exponentially) slow and, unlike in the case of the point-contact, its amount strongly depends on the choice of the qubit's initial state.

II. DESCRIPTION OF THE MODEL

Consider an electrostatic qubit interacting with a BEC consisting of N atoms initially trapped in the left well of a symmetric the double-well structure (Fig. 1b). The entire system can be described by the tunneling Hamiltonian $H = H_c + H_q + H_{\text{int}}$, where the three terms correspond to the condensate, the qubit and the interaction, respectively, and

$$H_c = E_0(c_L^\dagger c_L + c_R^\dagger c_R) - \Omega(c_L^\dagger c_R + c_R^\dagger c_L) \quad (1a)$$

$$H_q = E_1 d_1^\dagger d_1 + E_2 d_2^\dagger d_2 - \omega_0(d_1^\dagger d_2 + d_2^\dagger d_1) \quad (1b)$$

$$H_{\text{int}} = \delta\Omega d_2^\dagger d_2 (c_L^\dagger c_R + c_R^\dagger c_L). \quad (1c)$$

Here $c_{L,R}^\dagger$ is a boson creation operator in the left (right) reservoir, and $d_{1,2}^\dagger$ is the fermionic creation operator for the qubit. Tunneling between the reservoirs is suppressed when the electron is in the dot nearest to the barrier, so that $\delta\Omega = \Omega - \Omega' > 0$, Fig.1b.

For a static qubit trapped in one of the quantum dots, ($\omega_0 = 0$), each boson of the condensate oscillates between the reservoirs with the Rabi frequency Ω or Ω' . Thus, the probability of finding n bosons in the right reservoir at the time t is given by

$$P_n(t) = \binom{N}{n} \cos^{2(N-n)}(\Omega t) \sin^{2n}(\Omega t), \quad (2)$$

if the qubit's electron occupies the level E_1 of the nearer dot (or by the same expression with $\Omega \rightarrow \Omega'$ when it occupies the level E_2). In the interesting case when the tunneling rate for each atom is small but the number of atoms is large, we put $N \rightarrow \infty$, while $\sqrt{N}\Omega \rightarrow \kappa = \text{const}$, thus maintaining a finite current into the right reservoir. For small times, $t = \Delta t \ll \Omega^{-1}$ Eq. (2) yields $P_n(\Delta t) \approx (\kappa \Delta t)^{2n} e^{-(\kappa \Delta t)^2} / n!$ and, in particular, $P_0(\Delta t) \approx 1 - (\kappa \Delta t)^2$. Such a non-Markovian behavior of the BEC is in contrast with large fermion reservoirs (Fig. 1a), where one finds $1 - P_0(\Delta t) \propto (\Delta t)$, which is typical for a Markovian process.

Consider now the behavior of a dynamic qubit, $\omega_0 \neq 0$, subjected to a measurement with such a non-Markovian (BEC) detector. The wave function of the entire system can be written as $|\Psi(t)\rangle = \sum_{q,n} \psi_{qn}(t) |q\rangle |\phi_n\rangle$ where

$$|q\rangle |\phi_n\rangle \equiv d_q^\dagger |0\rangle_{\text{qub}} \frac{(c_L^\dagger)^{N-n} (c_R^\dagger)^n |0\rangle_{\text{cond}}}{\sqrt{(N-n)!n!}} \quad (3)$$

is the state corresponding the qubit localized in one of the quantum dots, $q = 1, 2$, and $n = 0, 1, 2, \dots$ is a number of bosons contained in the right well. For the corresponding probability amplitude $\psi_{qn}(t)$ we write $\psi_{qn}(t) = \langle q | \langle \phi_n | \exp(-Ht) | q_0 \rangle | \phi_0 \rangle$ state from which the reduced density matrix of the qubit is obtained by tracing out the BEC states,

$$\sigma_{qq'}(t) = \sum_n \psi_{qn}(t) \psi_{q'n}^*(t). \quad (4)$$

Putting, for convenience, $E_0 = 0$ one easily finds from Eqs. (1) that $H_q + H_{\text{int}}$ commutes with H_c and the evolution operator $U(t) \equiv \exp(-Ht)$ can be factorized,

$$U(t) = e^{-iH_c t} e^{-i(H_q + H_{\text{int}})t} \equiv U_c(t) U_{\text{qint}}(t). \quad (5)$$

Operators $U_c(t)$ and $U_{\text{qint}}(t)$ can be written in a simple form in basis of the eigenstates of the BEC Hamiltonian,

$$|\tilde{\phi}_n\rangle = \frac{(c_L^\dagger + c_R^\dagger)^{N-n} (c_L^\dagger - c_R^\dagger)^n |0\rangle}{\sqrt{2^N (N-n)!n!}}, \quad (6)$$

such that $H_c |\tilde{\phi}_n\rangle = (N-2n)\Omega |\tilde{\phi}_n\rangle$. Indeed, we have $\langle q, \tilde{\phi}_n | U_c(t) | q', \tilde{\phi}_{n'} \rangle = \exp[-i(N-2n)\Omega t] \delta_{nn'} \delta_{qq'}$ and also

$$\langle q | \langle \tilde{\phi}_n | U_{\text{qint}}(t) | q' \rangle | \tilde{\phi}_{n'} \rangle = \langle q | \hat{U}(t, \varepsilon_n) | q' \rangle \delta_{nn'}, \quad (7)$$

where $\varepsilon_n = (2n-N)\delta\Omega$. It is readily seen that $\hat{U}(t, \varepsilon_n) = \exp[-i(H_q + \varepsilon_n d_2^\dagger d_2)t]$ represents the evolution operator of an isolated *asymmetric* qubit with the level displacement $\varepsilon'_n = \varepsilon_n + E_2 - E_1$ (asymmetry parameter), whose matrix elements are easily found to be

$$\begin{aligned} \langle 1 | \hat{U}(t, \varepsilon_n) | 1 \rangle &= \left[\cos(\omega t) - i \frac{\varepsilon'_n}{2\omega} \sin(\omega t) \right] e^{-i\varepsilon'_n t/2} \\ \langle 2 | \hat{U}(t, \varepsilon_n) | 2 \rangle &= \left[\cos(\omega t) + i \frac{\varepsilon'_n}{2\omega} \sin(\omega t) \right] e^{-i\varepsilon'_n t/2} \\ \langle 1 | \hat{U}(t, \varepsilon_n) | 2 \rangle &= -i \frac{\varepsilon'_n}{\omega} \sin(\omega t) e^{-i\varepsilon'_n t/2} \end{aligned} \quad (8)$$

where $\omega \equiv \omega(\varepsilon_n) = \sqrt{(\varepsilon'_n/2)^2 + \omega_0^2}$ is the qubit's Rabi frequency and $\langle 2 | \hat{U}(t, \varepsilon_n) | 1 \rangle = \langle 1 | \hat{U}(t, \varepsilon_n) | 2 \rangle$.

For the reduced density matrix of the qubit in Eq. (4), with the help of Eqs. (4)-(8) we find

$$\sigma_{qq'}(t) = \sum_n \sigma_{qq'}(t, \varepsilon_n) |\langle \phi_0 | \tilde{\phi}_n \rangle|^2, \quad (9)$$

where

$$\sigma_{qq'}(t, \varepsilon_n) = \langle q | \hat{U}(t, \varepsilon_n) | q_0 \rangle \langle q_0 | \hat{U}^{-1}(t, \varepsilon_n) | q' \rangle. \quad (10)$$

is the density matrix corresponding to the unitary evolution of an isolated asymmetric qubit. With the initial state of the BEC given by $|\phi_0\rangle = (1/\sqrt{N!})(c_L^\dagger)^N |0\rangle$ we then find

$$|\langle \phi_0 | \tilde{\phi}_n \rangle|^2 = \frac{N!}{2^n n! (N-n)!} \simeq \frac{2}{\sqrt{2\pi N}} e^{-\frac{(N-2n)^2}{2N}} \quad (11)$$

where we have used the Sterling formula $K! \simeq \sqrt{2\pi K} K^K \exp(-K)$ to evaluate the factorials.

Now we assume that the qubit's coupling with each individual atom of the condensate ($\delta\Omega$) is weak, but its interaction with the entire condensate is considerable, and so is the variation of the BEC current ($\propto \sqrt{N} \delta\Omega$), induced by the qubit. Then taking the limit

$$N \rightarrow \infty, \quad \text{with} \quad \sqrt{N} \delta\Omega \rightarrow \alpha = \text{const} \quad (12)$$

we replace the sum over n in (9) by an integral, $\sum_n \rightarrow \int d\varepsilon/(2\delta\Omega)$. This yields

$$\sigma_{qq'}(t) = \int_{-\infty}^{\infty} \tilde{\sigma}_{qq'}(t, \varepsilon) \exp(-\varepsilon^2/2\alpha^2) \frac{d\varepsilon}{\sqrt{2\pi}\alpha} \quad (13)$$

where $\tilde{\sigma}_{qq'}(t, \varepsilon) \equiv [\sigma_{qq'}(t, \varepsilon) + \sigma_{qq'}(t, -\varepsilon)]/2$.

In the following we will consider only the case of a symmetric qubit, $E_1 = E_2$ [10]. Then for an initial qubit's state $|q_0\rangle = (a d_1^\dagger + b d_2^\dagger)|0\rangle_q$, we obtain from Eq. (10)

$$\tilde{\sigma}_{11}(t, \varepsilon) = |a|^2 + (|b|^2 - |a|^2)[1 - \cos(2\omega t)](\omega_0^2/2\omega^2) - \text{Im}(ab^*) \sin(2\omega t)(\omega_0/\omega), \quad (14a)$$

$$\tilde{\sigma}_{12}(t, \varepsilon) = i(|a|^2 - |b|^2) \sin(2\omega t)(\omega_0/2\omega) + ab^* \cos(2\omega t) + \text{Re}(ab^*)[1 - \cos(2\omega t)](\omega_0/\omega)^2, \quad (14b)$$

with $\omega \equiv \omega(\varepsilon) = \sqrt{(\varepsilon/2)^2 + \omega_0^2}$ and $\tilde{\sigma}_{22}(t) = 1 - \tilde{\sigma}_{11}(t)$, $\tilde{\sigma}_{21}(t) = \tilde{\sigma}_{12}^*(t)$.

III. DECOHERENCE OF QUBIT DUE TO ITS INTERACTION WITH THE BEC

The simple form of Eqs.(13)-(14) allows for an easy analysis of limiting cases. Indeed, the strength of interaction with the BEC, α , enters Eq.(13) only via the Gaussian cut-off factor $\exp(-\varepsilon^2/2\alpha^2)$. The factor determines the number of asymmetric configurations contributing of the qubit's evolution and, therefore the perturbation incurred upon the qubit by the BEC. (Note that when the interaction vanishes, $\alpha \rightarrow 0$, the Gaussian becomes narrow, and we recover the unperturbed evolution of the isolated qubit.)

It is readily seen that in the large time limit $t \rightarrow \infty$ the contributions for the rapidly oscillating terms in Eqs.(14a)-(14b) vanish. Evaluating the remaining integrals analytically shows that as $t \rightarrow \infty$ the density matrix of a qubit tends to a steady state σ^{st} given by

$$\begin{aligned} \sigma_{11}^{st} &= |a|^2 + (\sqrt{\pi}/2)z \exp(z^2) \text{erfc}(z) (|b|^2 - |a|^2), \\ \sigma_{12}^{st} &= \sqrt{\pi} z \exp(z^2) \text{erfc}(z) \text{Re}(ab^*), \end{aligned} \quad (15)$$

where $z = \sqrt{2}\omega_0/\alpha$ and $\text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty \exp(-t^2) dt$ is the complementary error function.

To evaluate the speed with which this steady state is attained we note that at large t the phase of the sines and cosines in Eqs.(14) develops a stationary region of

the width $\Delta\varepsilon = (4\omega_0/t)^{1/2}$ centered at $\varepsilon = 0$. Once $\Delta\varepsilon$ becomes small compared to the width of the Gaussian in Eq.(13), i.e. for $t \gg \omega_0/\alpha^2$, the contribution from the stationary region becomes proportional to $\Delta\varepsilon$ causing the time-dependent part of $\sigma_{qq'}$ in Eqs.(14) to decay as $1/\sqrt{t}$. (For a recent discussion of non-exponential decoherence behavior expected in other systems see, for example, Ref.[12, 13].) Explicitly, for $\omega_0 \neq 0$, the stationary phase method yields:

$$\begin{aligned} \sigma_{11}(t) &\simeq \sigma_{11}^{st} + \sqrt{\frac{\omega_0}{2\alpha^2 t}} \left[(|a|^2 - |b|^2) \cos\left(2\omega_0 t + \frac{\pi}{4}\right) \right. \\ &\quad \left. - 2 \text{Im}(ab^*) \sin\left(2\omega_0 t + \frac{\pi}{4}\right) \right] \end{aligned} \quad (16a)$$

$$\begin{aligned} \sigma_{12}(t) &\simeq \sigma_{12}^{st} + i\sqrt{\frac{\omega_0}{2\alpha^2 t}} \left[(|a|^2 - |b|^2) \sin\left(2\omega_0 t + \frac{\pi}{4}\right) \right. \\ &\quad \left. + 2 \text{Im}(ab^*) \sin\left(2\omega_0 t + \frac{\pi}{4}\right) \right], \end{aligned} \quad (16b)$$

where σ^{st} is given by Eq. (15). Figure 2 demonstrates that Eq.(16) (dot-dashed curve) coincides to graphical accuracy with the exact result (10) (solid curve) except at very short times.

Equations (15), (16) which describe the qubit's decoherence generated by the BEC employed as a measurement device represent our main result. The qubit's behavior is very different from that of a qubit interacting with electronic reservoirs[7], Fig. 1, or in a general with any Markovian environment, whose effect can be described by Bloch-like equations [14, 15, 16]. Indeed, it follows from Eqs. (16) that the relaxation to the final steady state is extremely *slow*, obeying the power law $\propto 1/\sqrt{t}$. One exception from this rule is a static qubit ($\omega_0 = 0$) for which the stationary region vanishes so that from Eqs. (13), (14), one easily obtains $\sigma_{12}(t) = ab^* \exp(-\alpha^2 t^2/2)$. In contrast, in a Markovian environment, a static or dynamic qubit undergoes an exponential relaxation to the final statistical mixture.

It also follows from Eq. (15) that, in general, the qubit's decoherence in the steady state is incomplete and its density matrix is not converted into a statistical mixture, $\sigma^{st} \neq \text{diag}(1/2, 1/2)$ as would be the case for a point-contact detector. Rather, complete decoherence is achieved only in the weak coupling limit ($\alpha \rightarrow 0$) [11] and only for the initial conditions corresponding to $\text{Re}(ab^*) = 0$. For a weak coupling, the dependence on the qubit's initial state can be understood in a following way. A real part of the qubit's off-diagonal density-matrix element $\text{Re} \sigma_{12}(t)$ can be written as $\text{Re} \sigma_{12}(t) = (1/2) \langle \Psi(t) | \hat{q}_+ | \Psi(t) \rangle$, where $\hat{q}_+ = d_1^\dagger d_2 + d_2^\dagger d_1$. If the qubit's levels are aligned ($E_1 = E_2$), the operator q_+ commutes with the total Hamiltonian, Eq. (1), in the limit of $\delta\Omega \rightarrow 0$. As a result $\text{Re} \sigma_{12}(t) \simeq \text{Re} \sigma_{12}(0)$. Therefore, the subspace of the qubit's states, corresponding to $\text{Im} \sigma_{12}(0) = 0$ is effectively *decoherence free* [17].

In strong coupling limit, $\alpha \rightarrow \infty$, in Eqs.(14a)-(14b) we only need to retain the terms which do not vanish for $|\varepsilon| \rightarrow \infty$, thus compensating for the factor α^{-1} in

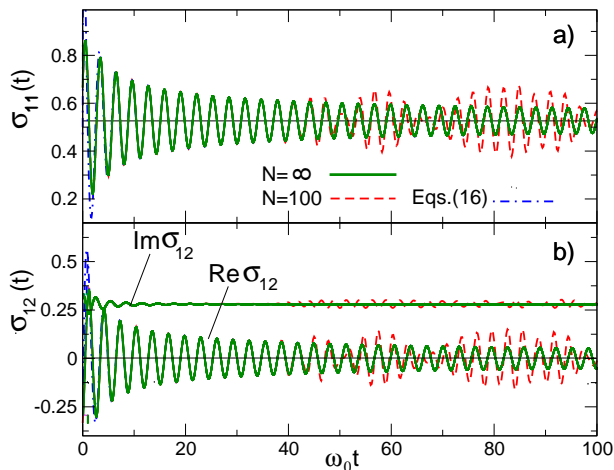


FIG. 2: (Color on line) a) The qubit's density matrix element σ_{11} vs. time t for $|q_0\rangle = [(1+i)|1\rangle + i|2\rangle]/\sqrt{3}$, $\alpha = 1$, $N \rightarrow \infty$, Eq.(13), (solid), $N = 100$, Eq.(9), (dashed) and the stationary phase approximation, Eqs. (16) (dash-dot). Horizontal line shows the large time asymptote (15) ; b) same as (a) but for σ_{12} .

Eq.(13). Accordingly, the off-diagonal density-matrix element would disappear at all times for all initial qubit's states, $\sigma_{12}(t) \rightarrow 0$. However, the result is not the statistical mixture, as in the case of weak coupling, but $\sigma(t) = \text{diag}(|a|^2, |b|^2)$. This corresponds to the so-called *pure dephasing* for a static ($\omega_0 = 0$) qubit [18] whose diagonal density-matrix elements remain constant while the off-diagonal elements vanish.

Finally, the finite size effects for a condensate with a large but finite number of atoms are shown Fig.2. These manifest themselves as an onset of irregular oscillations of the qubit's density matrix (dashed lines in Fig.2). The oscillations appear at times comparable with the Rabi period of an individual atom in the double well potential, $t \approx 2\pi\delta\Omega$, prior to which the qubit's evolution agrees with that in the presence of an infinite condensate. Mathematically, the effect occurs when the period of ever faster oscillating terms in Eqs.(14a)-(14b) becomes comparable with the separation between the energy levels of the condensate and Eq.(9) ceases to be a valid Riemann sum for the integral (13). Physically, the qubit begins to be affected by the size of the condensate at times of

the order the Poincare recurrence time of the latter, i.e., when the escape of the atoms into the right well can no longer be considered irreversible.

IV. SUMMARY

In summary, we have demonstrated that continuous monitoring of a qubit by a BEC produces a slow state-selective decoherence which obeys a power, rather than exponential, law in time (except for a static qubit, where the decoherence is extremely fast). Although this result was obtained in the limit $N \rightarrow \infty$, it can be confirmed by numerical evaluations of the qubit's density-matrix, Eq. (9) for a large but finite N , Fig. 2. It is this non-exponential relaxation and a strong dependence on the qubit's initial state that distinguishes the BEC model, with a single energy level in each of the reservoirs, from the exponential decoherence generated by a (Markovian) environment with a continuum spectrum of available states. Common to both environments is, however, freezing of the qubit's internal transitions in the strong interaction limit. This kind of Zeno effect [19, 20, 21] produced by the unitary evolution in the presence of an environment is somewhat different from its conventional prototype [22, 23] which arises from frequent observations of the evolving system. One remarkable feature of a Markovian environment is that the qubit's evolution under such frequent observations is practically indistinguishable from its unitary observation-free evolution [24]. For a qubit-BEC hybrid system whose behavior is explicitly non-Markovian, we expect the two types of evolution to be drastically different. A detailed investigation of this problem is, however, beyond the scope of the present paper.

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